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**Soliton equations in N-dimensions as exact reductions of Self-dual
Yang - Mills equation III. Soliton geometry in 2+1 and 3+1
dimensions¹**

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Abstract

Some aspects of the relation between differential geometry of curves and surfaces and multi-dimensional soliton equations are discussed. The connection between multi-dimensional soliton equations and the Self-dual Yang-Mills equation is considered.

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1 Introduction

One of the multidimensional integrable equations, important both in physics and mathematics, is the self-dual Yang-Mills (SDYM) equation [2, 25]. It arises in relativity [26, 3] and in field theory [27]. The SDYM equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space. Integrability for a SDYM connection means that its curvature vanishes on certain two-planes in the tangent space of the Grassmannian. As shown in [4,5], This allows one to characterize SDYM connections in terms of the splitting problem for a transition function in a holomorphic bundle over the Riemann sphere, i.e. the trivialization of the bundle [28, 29].

It is well known that the SDYM equation admits reductions [2, 4-5, 18-19] (see also the book [1] and references therein) to many soliton equations in 1+1 and 2+1 dimensions, such as: i) the Boussinesq, Korteweg-de Vries (KdV) and modified KdV, nonlinear Schrodinger, N-wave, Sine-Gordon, Ernst and chiral field equations in 1+1 dimensions; ii) the Kadomtsev - Petviashvili (KP), Davey - Stewartson (DS), Bogomolny, Toda molecule and the Ward (chiral field with torsion) equations, in 2+1 dimensions. Moreover, all integrable systems are may be some reductions of the SDYM equation or its generalizations (see, e.g. [2, 30]). It is interesting to note that the SDYM equation contains also some known ordinary differential equations (ODE) as one-dimensional reductions [1]. These ODE include: the Euler-Arnold top, Kovalevskaya tops, Nahm, Chazy and Painleve equations. Also we note that between the Einstein and Yang-Mills equations as well exist some connections [3].

On the other hand, it is well known that there exist some another equations - the mM-LXII and M-LXII equations which contains many (may be all) soliton equations in 1+1, 2+1 and 3+1 dimensions, including above mentioned. These equations are the integrability conditions of the system describing the moving orthogonal trihedral of a curve or surface [8].

First aim of the present paper is to show that soliton equations in 2+1 and 3+1 dimensions as well as their "master equations" - the mM-LXII and M-LXII equations are exact reductions of "premaster equation" - the SDYM equation. We do this by considering the SDYM equation for $GL(Z, C)$ group ($Z=2,3$).

The other question which we will discuss below is the relation between differential geometry (curves, surfaces) and multidimensional soliton equations. Commonly known fact: between geometry and integrable systems exists a

deeper connection [8-17, 20-22]. Classical examples demonstrating such connections are: the Liouville equation which describes minimal surfaces in E^3 and the sine-Gordon equation which encodes the whole geometry of the pseudospherical surfaces, i.e. surfaces with negative constant Gaussian curvature. The discovery of the Inverse Scattering Transform (IST) method [31] inspired a lot of modern work on the study of the connection between geometry and integrable systems. So the discussion some aspects of a nature of the tandem "geometry-soliton" in 2+1 and 3+1 dimensions is second aim of this paper. We feel our older and permanent love to spin systems (Heisenberg magnet models = Landau-Lifshitz equations) and below we try consider to some underlying problems from our "spin" point of view, that is, third aim of this note.

2 The (1+1)-dimensional equations of curves/surfaces

In this section we review briefly the some known basic facts from 1+1 dimensions to set our notations and terminology. Consider the curve in 3-dimensional space or two-dimensional surface. Equations of such curves and surfaces, following [8] we can write in the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (1a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (1b)$$

where

$$C = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta \sigma & -\tau & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix} \quad (2)$$

Here \mathbf{e}_j is the moving trihedral of a curve or surface,

$$\mathbf{e}_1^2 = \beta = \pm 1, \quad \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1 \quad (3)$$

k, σ and τ are called the normal curvature, geodesic curvature, and geodesic torsion, respectively. So we have

$$C_t - G_x + [C, G] = 0 \quad (4)$$

In the curves case, the equation (1a) is as $\sigma = 0$ the Serret - Frenet equation (SFE). In surface interpretation, (1) is the Gauss-Weingarden equation (GWE) while (4) is the Codazzi-Mainardi-Peterson equation (CMPE) in orthogonal basis. In the latter case, $k, \tau, \sigma, \omega_j$ are some functions of E, F, G, L, M, N (coefficients of first and second fundamental forms). Let us rewrite the equation (4) in the form

$$U_t - V_x + [U, V] = 0 \quad (5)$$

where

$$U = \frac{1}{2i} \begin{pmatrix} \tau & k - i\sigma \\ k + i\sigma & -\tau \end{pmatrix}, \quad V = \frac{1}{2i} \begin{pmatrix} \omega_1 & \omega_3 - i\omega_2 \\ \omega_3 + i\omega_2 & -\omega_1 \end{pmatrix} \quad (6)$$

The Lax representation (LR) of the equation (4)-(5) is given by

$$\psi_x = U\psi \quad (7a)$$

$$\psi_t = V\psi \quad (7b)$$

Usually, we assume that [8]

$$k = \sum_j \lambda^j k_j, \quad \tau = \sum_j \lambda^j \tau_j, \quad \sigma = \sum_j \lambda^j \sigma_j, \quad \omega_k = \sum_j \lambda^j \omega_{kj} \quad (8)$$

where $j = 0, \pm 1, \pm 2, \pm 3, \dots$. Sometimes, instead of (8) we take the following more general case

$$k = \sum_j h_{1j} k_j, \quad \tau = \sum_j h_{2j} \tau_j, \quad \sigma = \sum_j h_{3j} \sigma_j, \quad \omega_k = \sum_j h_{4j} \omega_{kj} \quad (9)$$

Here $h_{ij} = h_{ij}(\lambda)$ and λ is some characteristic parameter of curves or surfaces or some function of such parameters. In the cases (8) and (9), i.e when $k, \tau, \sigma, \omega_j$ are some functions of λ , the equations (1), (4)=(5) and (7), we call the M-LXIX, M-LXX and M-LXXI equations respectively. [These conditional notations we use in order to accurately distinguish these equations from the case when $k, \tau, \sigma, \omega_j$ are independent of λ and for convenience in our internal working kitchen. The more so, as of course it is not means that we have e.g. 72 equations.] Particular cases.

i) *The Gauss-Weingarden and Codazzi-Mainardi-Peterson equations.* These equations correspond to the case

$$k = k_0, \quad \tau = \tau_0, \quad \sigma = \sigma_0, \quad \omega_k = \omega_{k0} \quad (10)$$

ii) *The ZS-AKNS problem.* The famous Zakharov- Shabat-Ablovitz-Kaup-Newell-Segur (ZS-AKNS) spectral problem which generate many soliton equations in 1+1 is the particular case of the M-LXXI equation (7a) as

$$k_0 = i(p + q), \quad k_j = 0, \quad j \neq 0 \quad (11a)$$

$$\sigma_0 = p - q, \quad \sigma_j = 0, \quad j \neq 0 \quad (11b)$$

$$\tau_1 = -2, \quad \tau_j = 0, \quad j \neq 1 \quad (11c)$$

iii) *The Kaup-Newell-Wadati-Konno-Ichicawa spectral problem.* This case corresponds to reduction

$$k_1 = i\lambda(p + q), \quad k_j = 0, \quad j \neq 1 \quad (11d)$$

$$\sigma_1 = \lambda(p - q), \quad \sigma_j = 0, \quad j \neq 1 \quad (11e)$$

$$\tau_1 = -2, \quad \tau_j = 0, \quad j \neq 1 \quad (11f)$$

iv) *The equations of a principal chiral field.* The equations of a principal chiral field for functions u, v

$$u_t + \frac{1}{2}[u, v] = 0, \quad v_x - \frac{1}{2}[u, v] = 0 \quad (11g)$$

can equally be represented with the aid of (4) or (5), if we choose in this case

$$h_{11} = \frac{1}{1-\lambda}, \quad k_j = \tau_j = \sigma_j = 0, \quad j \neq 1 \quad (11h)$$

$$U = \frac{u}{1-\lambda}, \quad V = \frac{v}{\lambda+1} \quad (11i)$$

v) *The (1+1)-dimensional mM-LXVI equation.* This case corresponds to the reduction when

$$k^2 + \tau^2 + \sigma^2 = n^2 \quad (11j)$$

So we see that practically all (1+1)-dimensional soliton equations can be obtained from the M-LXX equation (4)=(5) as some reductions.

3 Curves and Surfaces in 4-dimensions: the mM-LXI equation

3.1 The (3+1)-dimensional mM-LXI equation: the space case

The (3+1)-dimensional mM-LXI equation reads as [8]

$$\begin{aligned} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x &= A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \\ \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_z &= C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \end{aligned} \quad (12)$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta \sigma & -\tau & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 0 & n_3 & -n_2 \\ -\beta n_3 & 0 & n_1 \\ \beta n_2 & -n_1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix}. \end{aligned} \quad (13)$$

3.2 The (3+1)-dimensional modified M-LXII equation: the space case

From (12), we obtain the following mM-LXII equation [8]

$$A_y - B_x + [A, B] = 0 \quad (14a)$$

$$A_z - C_x + [A, C] = 0 \quad (14b)$$

$$A_t - D_x + [A, D] = 0 \quad (14c)$$

$$B_z - C_y + [B, C] = 0 \quad (14d)$$

$$B_t - D_y + [B, D] = 0 \quad (14e)$$

$$C_t - D_z + [C, D] = 0 \quad (14f)$$

3.2.1 Lax representation of the (3+1)-dimensional mM-LXII equation

The mM-LXII equation (14) we can rewrite in the following form

$$F_{xy} = U_y - V_x + [U, V] = 0 \quad (15a)$$

$$F_{xz} = U_z - W_x + [U, W] = 0 \quad (15b)$$

$$F_{xt} = U_t - T_x + [U, T] = 0 \quad (15c)$$

$$F_{yz} = V_z - W_y + [V, W] = 0 \quad (15d)$$

$$F_{yt} = V_t - T_y + [V, T] = 0 \quad (15e)$$

$$F_{zt} = W_t - T_z + [W, T] = 0 \quad (15f)$$

where

$$U = \frac{1}{2i} \begin{pmatrix} \tau & k - i\sigma \\ k + i\sigma & -\tau \end{pmatrix}, \quad T = \frac{1}{2i} \begin{pmatrix} \omega_1 & \omega_3 - i\omega_2 \\ \omega_3 + i\omega_2 & -\omega_1 \end{pmatrix}$$

$$V = \frac{1}{2i} \begin{pmatrix} m_1 & m_3 - im_2 \\ m_3 + im_2 & -m_1 \end{pmatrix}, \quad W = \frac{1}{2i} \begin{pmatrix} n_1 & n_3 - in_2 \\ n_3 + in_2 & -n_1 \end{pmatrix} \quad (16)$$

The LR of the mM-LXII equation (14) or (15) is given by

$$g_x = Ug, \quad g_y = Vg, \quad g_z = Wg, \quad g_t = Tg \quad (17)$$

We note that this LR can be rewritten in the more usual form

$$Lg = 0, \quad Mg = 0 \quad (18)$$

where, for example, L, M we can take in the form

$$L = \partial_x + a\lambda\partial_y + b\lambda^2\partial_z - (U + a\lambda V + bW), \quad (19a)$$

$$M = \partial_t + e\lambda^3\partial_y + f\lambda^4\partial_z - (T + e\lambda^3V + f\lambda^4W) \quad (19b)$$

or

$$L = D_x + a\lambda D_y + b\lambda^2 D_z, \quad M = D_t + e\lambda^3 D_y + f\lambda^4 D_z \quad (20)$$

Here D_j are the covariant derivatives

$$D_x = \partial_x - U, \quad D_y = \partial_y - V, \quad D_z = \partial_z - W, \quad D_t = \partial_t - T \quad (21)$$

3.3 The (3+1)-dimensional mM-LXI equation: the plane case

The mM-LXI equation in plane has the form [8]

$$\begin{aligned} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_x &= A_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_y = B_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \\ \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_z &= C_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_t = D_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \end{aligned} \quad (22)$$

where

$$\begin{aligned} A_p &= \begin{pmatrix} 0 & k \\ -\beta k & 0 \end{pmatrix}, \quad B_p = \begin{pmatrix} 0 & m_3 \\ -\beta m_3 & 0 \end{pmatrix} \\ C_p &= \begin{pmatrix} 0 & n_3 \\ -\beta n_3 & 0 \end{pmatrix}, \quad D_p = \begin{pmatrix} 0 & \omega_3 \\ -\beta \omega_3 & 0 \end{pmatrix}. \end{aligned} \quad (23)$$

3.4 The (3+1)-dimensional modified M-LXII equation: the plane case

In the plane case the mM-LXII equation takes the following simple form

$$k_y = m_{3x}, \quad k_z = n_{3x}, \quad m_{3z} = n_{3y}, \quad m_{3t} = \omega_{3y}, \quad n_{3t} = \omega_{3z} \quad (24a)$$

Hence we get

$$m_3 = \partial_x^{-1} k_y, \quad n_3 = \partial_x^{-1} k_z \quad (24b)$$

The (3+1)-dimensional nonlinear evolution equation for k has the form

$$k_t = \omega_{3x} \quad (25)$$

We note that this equation gives the (3+1)-dimensional KdV, KP, NV, mNV and so on, equations as some reductions with the corresponding LR (17).

3.5 The M-LXI and M-LXII equations

The M-LXI and M-LXII equations are the particular cases of the mM-LXI and mM-LXII equations as $\sigma = 0$ respectively.

4 Curves and Surfaces in 4 dimensions: the modified M-LXVIII equation

4.1 The modified M-LXVIII equation: the space case

One of important four dimensional equation of curves and/or surfaces is the following modified M-LXVIII (mM-LXVIII) equation

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = a \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_3} + C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (26a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = b \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} + G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (26b)$$

where ξ_j are may be real or complex coordinates, a, b are some parameters,

$$C = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta \sigma & -\tau & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix} \quad (27)$$

The compativility condition of the equations (26) is given by

$$C_{0\xi_2} - bC_{0\xi_4} + aG_{1\xi_3} - G_{1\xi_1} + [C_0, G_1] = 0 \quad (28)$$

The LR of this equation can be written as

$$\psi_{\xi_1} = a\psi_{\xi_3} + B_0g \quad (29a)$$

$$\psi_{\xi_2} = b\psi_{\xi_4} + B_1g \quad (29b)$$

Hence we get

$$B_{0\xi_2} - bB_{0\xi_4} + aB_{1\xi_3} - B_{1\xi_1} + [B_0, B_1] = 0 \quad (30)$$

with

$$B_0 = \frac{1}{2i} \begin{pmatrix} \tau & k - i\sigma \\ k + i\sigma & -\tau \end{pmatrix}, \quad B_1 = \frac{1}{2i} \begin{pmatrix} \omega_1 & \omega_3 - i\omega_2 \\ \omega_3 + i\omega_2 & -\omega_1 \end{pmatrix} \quad (31)$$

4.2 The mM-LXVIII equation: the plane case

The mM-LXVIII equation in the plane case has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_{\xi_1} = a \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_{\xi_3} + C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \quad (32a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_{\xi_2} = b \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_{\xi_4} + G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \quad (32b)$$

4.3 The M-LXVIII equation

The M-LXVIII equation is the particular case of the mM-LXVIII equation as $\sigma = 0$.

5 SDYM equation

It is standard to define a Yang-Mills vector bundle over a four-dimensional manifold M with connection one-form $A = A_\mu(x^\nu)dx^\mu$. The SDYM equations in this manifolds let us write to set our notation:

$$F = *F \quad (33)$$

where F is a curvature 2-form pulled back to M from the gauge bundle $P(M, g)$, explicitly:

$$F = d\omega + \omega \wedge \omega \quad (34)$$

Here the connection 1-form ω on P takes values in the Lie algebra g of the gauge group G . In terms of Cartesian coordinates x^μ , they can be expressed as

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\sigma}F_{\kappa\sigma} \quad (35)$$

where $\mu, \nu, \dots = 1, 2, 3, 4$, $\epsilon_{\mu\nu\kappa\sigma}$ stands for the completely antisymmetric tensor in four dimensions with the convention: $\epsilon_{1234} = 1$. The components of the field strength ($F_{\mu\nu}$) are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (36)$$

We note that the SDYM equations (33) are invariant under the gauge transformation

$$A_\mu \rightarrow \phi^{-1} A_\mu \phi - \phi^{-1} \partial_\mu \phi \quad (37)$$

Let us introduce the some null coordinates ξ_j in which the metric on M is $ds^2 = d\xi_1 d\xi_3 + d\xi_2 d\xi_4$. For these coordinates, the SDYM equations are given by

$$F_{\xi_1 \xi_2} = 0 \quad (38a)$$

$$F_{\xi_3 \xi_4} = 0 \quad (38b)$$

$$F_{\xi_1 \xi_4} - F_{\xi_2 \xi_3} = 0 \quad (38c)$$

or

$$\partial_{\xi_1} A_2 - \partial_{\xi_2} A_1 + [A_1, A_2] = 0 \quad (39a)$$

$$\partial_{\xi_3} A_4 - \partial_{\xi_4} A_3 + [A_3, A_4] = 0 \quad (39b)$$

$$\partial_{\xi_1} A_4 - \partial_{\xi_4} A_1 + [A_1, A_4] = \partial_{\xi_2} A_3 - \partial_{\xi_3} A_2 + [A_2, A_3] \quad (39c)$$

For the equations (38)-(39), the LR has the form

$$L\Phi = 0, \quad M\Phi = 0 \quad (40)$$

where

$$L = D_{\xi_1} - \lambda D_{\xi_3}, \quad M = D_{\xi_2} - \lambda D_{\xi_4} \quad (41)$$

and λ is the spectral parameter.

6 The 4-dimensional mM-LXVIII equation and SDYM equation

To establish the connection between the 4-dimensional mM-LXVIII equation and SDYM equation, we rewrite the LR (29) of the mM-LXVIII equation in the form

$$L\psi = 0, \quad M\psi = 0 \quad (42)$$

where

$$L = D_{\xi_1} - aD_{\xi_3}, \quad M = D_{\xi_2} - bD_{\xi_4} \quad (43)$$

Here

$$D_{\xi_j} = \partial_{\xi_j} - A_j, \quad A_1 - aA_3 = B_0, \quad A_2 - bA_4 = B_1 \quad (44)$$

We see that the equations (40) and (42) have the same form if $a = b = \lambda$. So the 4-dimensional mM-LXVIII equation (28)=(30) is equivalent to the SDYM equations (38)=(39) as $a = b = \lambda$. Hence follows that the curve or surfaces (26) is integrable in this case and in this sense, i.e. they are the integrable curve or integrable surface in four ($4=3+1=2+2=1+3$) dimensions.

7 The 4-dimensional mM-LXII equations and SDYM equation

To establish the connection between the mM-LXII equation (14)=(15) and the SDYM equation (38)=(39) we consider the following transformation for coordinates

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = H \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad (45)$$

where

$$H = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad (46)$$

Also we use the expressions for U, V, W, T

$$U = b_{11}A_{\xi_1} + b_{12}A_{\xi_2} + b_{13}A_{\xi_3} + b_{14}A_{\xi_4} \quad (47a)$$

$$V = b_{21}A_{\xi_1} + b_{22}A_{\xi_2} + b_{23}A_{\xi_3} + b_{24}A_{\xi_4} \quad (47b)$$

$$W = b_{31}A_{\xi_1} + b_{32}A_{\xi_2} + b_{33}A_{\xi_3} + b_{34}A_{\xi_4} \quad (47c)$$

$$T = b_{41}A_{\xi_1} + b_{42}A_{\xi_2} + b_{43}A_{\xi_3} + b_{44}A_{\xi_4} \quad (47d)$$

We can consider the other may be more general transformation

$$\xi_j = f_j(x, y, z, t) \quad (48)$$

where f_j are some functions. Then after some algebra we have

$$F_{xy} = l_{12}^1 F_{\xi_1 \xi_3} + l_{12}^2 F_{\xi_2 \xi_4} + l_{12}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (49a)$$

$$F_{xz} = l_{13}^1 F_{\xi_1 \xi_3} + l_{13}^2 F_{\xi_2 \xi_4} + l_{13}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (49b)$$

$$F_{xt} = l_{14}^1 F_{\xi_1 \xi_3} + l_{14}^2 F_{\xi_2 \xi_4} + l_{14}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (49c)$$

$$F_{yz} = l_{23}^1 F_{\xi_1 \xi_3} + l_{23}^2 F_{\xi_2 \xi_4} + l_{23}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (49d)$$

$$F_{yt} = l_{24}^1 F_{\xi_1 \xi_3} + l_{24}^2 F_{\xi_2 \xi_4} + l_{24}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (49e)$$

$$F_{zt} = l_{34}^1 F_{\xi_1 \xi_3} + l_{34}^2 F_{\xi_2 \xi_4} + l_{34}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (49f)$$

So if $F_{\mu\nu}(\mu, \nu = \xi_1, \xi_2, \xi_3, \xi_4)$ satisfy the SDYM equation (38) then $F_{ij}(i, j = x, y, z, t)$ satisfy the mM-LXII equation (15).

8 The (2+1)-dimensional mM-LXVIII equation and SDYM equation

The mM-LXVIII equation in 3=2+1 dimensions can be written for example in the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (50a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = b \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} + G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (50b)$$

In this case the corresponding LR takes the form

$$\psi_{\xi_1} = B_0 g \quad (51a)$$

$$\psi_{\xi_2} = b\psi_{\xi_4} + B_1 g \quad (51b)$$

Hence we get

$$B_0 \xi_2 - bB_0 \xi_4 - B_1 \xi_1 + [B_0, B_1] = 0 \quad (52)$$

The corresponding (2+1)-dimensional SDYM equations have the forms

$$\partial_{\xi_1} A_2 - \partial_{\xi_2} A_1 + [A_1, A_2] = 0 \quad (53a)$$

$$\partial_{\xi_4} A_3 - [A_3, A_4] = 0 \quad (53b)$$

$$\partial_{\xi_1} A_4 - \partial_{\xi_4} A_1 + [A_1, A_4] = \partial_{\xi_2} A_3 + [A_2, A_3] \quad (53c)$$

9 The (2+1)-dimensional mM-LXI and mM-LXII equations: the space case and SDYM equation

9.1 The (2+1)-dimensional mM-LXI equation for the space case

The (2+1)-dimensional mM-LXI equation reads as

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (54a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (54b)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (54c)$$

9.2 The (2+1)-dimensional mM-LXII equation: the space case

From (54), we obtain the following mM-LXII equation [8]

$$A_y - B_x + [A, B] = 0 \quad (55a)$$

$$A_t - D_x + [A, D] = 0 \quad (55b)$$

$$B_t - D_y + [B, D] = 0 \quad (55c)$$

Some (2+1)-dimensional soliton equations such as: DS, Zakharov, (2+1)-complex mKdV, (2+1)-dNLS and so on, are exact reductions of the mM-LXII equation (55).

9.3 Lax representation of the (2+1)-dimensional mM-LXII equation

As above the mM-LXII equation (55) we write in the following form

$$F_{xy} = U_y - V_x + [U, V] = 0 \quad (56a)$$

$$F_{xt} = U_t - T_x + [U, T] = 0 \quad (56b)$$

$$F_{yt} = V_t - T_y + [V, T] = 0 \quad (56c)$$

So that the LR of the mM-LXVIII equation (55)=(56) is given by

$$g_x = U g, \quad g_y = V g, \quad g_t = T g \quad (57)$$

We note that this LR we can write in the more usual form

$$Lg = 0, \quad Mg = 0 \quad (58)$$

where, for example, L, M we can take in the form

$$L = \partial_x + a\lambda\partial_y - (U + a\lambda V), \quad (59a)$$

$$M = \partial_t + e\lambda^2\partial_y - (T + e\lambda^2 V) \quad (59b)$$

or

$$L = D_x + a\lambda D_y, \quad M = D_t + e\lambda^3 D_y. \quad (60)$$

9.4 The (2+1)-dimensional mM-LXII equation and SDYM equation

To establish the connection between the mM-LXII equation (55)=(56) and the SDYM equation (38) we consider as above the following transformation for coordinates

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = H \begin{pmatrix} x \\ y \\ 0 \\ t \end{pmatrix} \quad (61)$$

where

$$H = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad (62)$$

Also we use the expressions for U, V, W, T

$$U = b_{11}A_{\xi_1} + b_{12}A_{\xi_2} + b_{13}A_{\xi_3} + b_{14}A_{\xi_4} \quad (63a)$$

$$V = b_{21}A_{\xi_1} + b_{22}A_{\xi_2} + b_{23}A_{\xi_3} + b_{24}A_{\xi_4} \quad (63b)$$

$$T = b_{41}A_{\xi_1} + b_{42}A_{\xi_2} + b_{43}A_{\xi_3} + b_{44}A_{\xi_4} \quad (63c)$$

We can consider the other may be more general transformation for coordinates

$$\xi_j = f_j(x, y, t) \quad (64)$$

Then after some algebra we have

$$F_{xy} = l_{12}^1 F_{\xi_1 \xi_3} + l_{12}^2 F_{\xi_2 \xi_4} + l_{12}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (65a)$$

$$F_{xt} = l_{14}^1 F_{\xi_1 \xi_3} + l_{14}^2 F_{\xi_2 \xi_4} + l_{14}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (65b)$$

$$F_{yt} = l_{24}^1 F_{\xi_1 \xi_3} + l_{24}^2 F_{\xi_2 \xi_4} + l_{24}^3 (F_{\xi_1 \xi_2} - F_{\xi_3 \xi_4}) \quad (65c)$$

So if $F_{\mu\nu}(\mu, \nu = \xi_1, \xi_2, \xi_3, \xi_4)$ satisfy the SDYM equation (38) then $F_{ij}(i, j = x, y, t)$ satisfy the mM-LXII equation (55)=(56).

10 The (2+1)-dimensional mM-LXI and mM-LXII equations: the plane case

10.1 The (2+1)-dimensional mM-LXI equation: the plane case

The (2+1)-dimensional mM-LXI equation in plane has the form [8]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_x = A_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_y = B_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_t = D_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \quad (66)$$

10.2 The (2+1)-dimensional mM-LXII equation: the plane case

In the plane case the mM-LXII equation takes the following simple form

$$k_y = m_{3x}, \quad m_{3t} = \omega_y \quad (67a)$$

Hence we get

$$m_3 = \partial_y^{-1} k_y \quad (67b)$$

The nonlinear evolution equation has the form

$$k_t = \omega_{3x} \quad (68)$$

Many (2+1)-dimensional integrable equations such as the Kadomtsev-Petviashvili, Novikov-Veselov (NV), mNV, KNV, (2+1)-KdV, mKdV equations are the integrable reductions of the M-LXII equation (68).

11 The (2+1)-dimensional M-LXI and M-LXII equations

The (2+1)-dimensional M-LXI and M-LXII equations are the particular cases of the (2+1)-dimensional mM-LXI and mM-LXII equations as $\sigma = 0$ respectively.

12 The (2+1)-dimensional mM-LXII equation as the reduction of SDYM equation

One of a way to establish the connection between the mM-LXII equation (55)-(56), i.e. soliton equations in 2+1 and the SDYM equation (38) is the following. Consider the coordinates

$$\xi_1 = it, \quad \xi_2 = -it, \quad \xi_3 = x + iy, \quad \xi_4 = x - iy, \quad (69)$$

Now in the SDYM equation (38)=(39) we take

$$A_1 = -iD, \quad A_2 = iD, \quad A_3 = A - iB, \quad A_4 = A + iB. \quad (70)$$

where we mention that A, B, C, D are in our case real matrices. Then the SDYM equation (39) reduces to the (2+1)-dimensional mM-LXII equation (56). So the (2+1)-dimensional mM-LXII equation is the integrable reduction of the SDYM equation.

13 Soliton equations in 2+1 dimensions are exact reductions of SDYM equation

As many (may be all) soliton equations in 2+1 dimensions are some integrable reductions of the mM-LXII and/or M-LXII equations (55) and/or (56) then as follows from the results of the previous section these (2+1)-dimensional soliton equations are exact reductions of the SDYM equation (see, e.g [18, 22]).

14 The mM-LXII and Soliton equations in 3+1 dimensions as exact reductions of SDYM equation

Here we have the same arguments as in sections 12 and 13 [19, 22].

15 Surfaces with the restriction: $k^2 + \tau^2 + \sigma^2 = n^2(x, y, z, t)$. The M-LXVI equation and spin systems. L-equivalence and G-equivalence

Until now we not spoken about spin systems although not forget their. They are to us something greater than only mathematical object. So that let us find the place for them in our formalism and in this note. To this purpose,

we remember the M-LXVI equation which is the particular case of some above considered equations as

$$k^2 + \tau^2 + \sigma^2 = n^2(x, y, z, t) \quad (71)$$

Below we consider the case when $n = \text{constant}$. Let

$$k = nS_1, \quad \sigma = nS_2, \tau = nS_3 \quad (72)$$

Then from (71) follows that

$$S_1^2 + S_2^2 + S_3^2 = 1 \quad (73)$$

Consider the simplest example, the (1+1)-dimensional M-LXVI equation having the form

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_x = C' \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \quad (74a)$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_t = G' \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \quad (74b)$$

where

$$C' = n \begin{pmatrix} 0 & S_1 & -S_2 \\ -\beta S_1 & 0 & S_3 \\ \beta S_2 & -S_3 & 0 \end{pmatrix}, \quad G' = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta\omega_3 & 0 & \omega_1 \\ \beta\omega_2 & -\omega_1 & 0 \end{pmatrix} \quad (75)$$

Here \mathbf{f}_j is the new moving trihedral of a curve or surface,

$$\mathbf{f}_1^2 = \beta = \pm 1, \quad \mathbf{f}_2^2 = \mathbf{f}_3^2 = 1 \quad (76)$$

and ω_j are already some functions of S_j, S_{kx}, n . So we have the (1+1)-dimensional M-0 equation

$$C'_t - G'_x + [C', G'] = 0 \quad (77)$$

If remember that $k, \tau, \sigma, \omega_j$ are some functions of E, F, G, L, M, N (coefficients of first and second fundamental forms) then the M-LXVI and M-0 equations (74) and (77) describe some special class of surfaces with the some restriction for E, F, G, L, M, N which follows from (71). Let us rewrite the M-0 equation (77) in the form

$$S_t - \frac{1}{n}V_x + [S, V] = 0 \quad (78)$$

where

$$V = \frac{1}{2i} \begin{pmatrix} \omega_1 & \omega_3 - i\omega_2 \\ \omega_3 + i\omega_2 & -\omega_1 \end{pmatrix} \quad (79)$$

The LR of the equation (78) is given by

$$\psi_x = nS\psi \quad (80a)$$

$$\psi_t = V\psi \quad (80b)$$

If we take

$$V = -inSS_x - 2in^2S \quad (81)$$

then from the equation (78) we obtain the isotropic Landau-Lifshitz equation (LLE)

$$2iS_t = [S, S_{xx}] \quad (82)$$

where n plays the role of spectral parameter. Finally we consider the gauge transformation

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = E \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (83)$$

Then hence and from (1) and (74) we get

$$C = E^{-1}C'E - E^{-1}E_x, \quad G = E^{-1}G'E - E^{-1}E_t, \quad (84)$$

and the M-LXX equation give rise to the NLSE. In this case we speak that the NLSE is L-equivalent (Lakshmanan equivalent) counterpart of the LLE [9]. As known between these equations take place G-equivalence (gauge equivalence) [24]. The other reductions of the M-LXVI and M-0 equations, including and multi-dimensional spin systems and for details, see e.g. [22].

16 Summary

In this paper, we have considered some aspects of the relation between differential geometry of curves and surfaces and multidimensional soliton equations. Our approach permits find some integrable classess of curves and surfaces (soliton geometry). Starting from the known fact that many soliton equations, for example, in 2+1 dimensions are particular cases of the mM-LXII (and/or M-LXII) equation, we have shown they are exact reductions of the SDYM equation. The connection between spin systems and curves/surfaces is also discussed. Finally we note that the more detailed presentation of the above discussed results were given in our previous notes (see, e.g. [18, 19, 22]).

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References

- [1] Ablowitz M J and Clarkson P A 1992 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (LMS Lecture Note Series 149) (Cambridge: Cambridge University Press)

- [2] Ward R S 1985 *Phil. Trans. R. Soc. A* **315** 451-7
- [3] Mason L J and Newman E T 1989 *Commun. Math. Phys.* **121** 659-68
- [4] Mason L J and Sparling G A J 1989 *Phys. Lett.* **137A** 29-33
- [5] Legare M 1995 "Symmetry reductions of the Lax pair of the four-dimensional euclidean self-dual Yang-Mills equations" *hep-th/9508093*
- [6] Guil F and Manas M 1993 "Two-dimensional integrable systems and self-dual Yang-Mills equations" *hep-th/9307021*
- [7] Szmigielski J 1993 " On soliton content of Self-dual Yang-Mills equations" *hep-th/9311119*
- [8] Myrzakulov R 1987 On some integrable and nonintegrable soliton equations of magnets *Preprint* (Alma-Ata: HEPI)
- [9] Lakshmanan M 1977 *Phys. Lett.* **61A** 53
- [10] Sym A 1991 *Springer Lecture Notes in Physics*, ed. by R. Martini (Springer, Berlin, 1985), **239** 154
- [11] Doliwa A and Santini P M 1994 *Phys. Lett. A* **185** 373
- [12] Sasaki R 1979 *Nucl. Phys.* **B154** 343
- [13] Konopelchenko B G 1993 *Preprint* INP 93-114
- [14] Taimanov I A 1995 *Preprint dg-ga/9511005*
- [15] Balakrishnan R, Bishop A R and Dandoloff R 1993 *Phys. Rev. B* **47** 5438
- [16] Bobenko A I 1994 "Surfaces in terms of 2 by 2 matrices. Old and New integrable systems *Lect. Notes in Physics*, **239** 154
- [17] Nakayama K, Hoppe J and Wadati M 1995 "On the Level-Set Formulation of Geometrical Models" *J. Phys. Soc. Jpn.* **64** 403-407
- [18] Myrzakulov R and Myrzakul Kur. R 1994 "Soliton equations in N-dimensions as exact reductions of Self-dual Yang-Mills equation I. The N=2+1 case." *Preprint CNLP-1994-12* (Alma-Ata, CNLP)
- [19] Myrzakulov R and Myrzakul Kur. R 1995 "Soliton equations in N-dimensions as exact reductions of Self-dual Yang-Mills equation II. The N=3+1 case." *Preprint CNLP-1995-09* (Alma-Ata, CNLP)
- [20] Myrzakulov R 1994 "Soliton equations in 2+1 dimensions and Differential geometry of curves/surfaces" *Preprint CNLP-1994-02* (Alma-Ata, CNLP)
- [21] Myrzakulov R and Lakshmanan M 1996 "On the geometrical and gauge equivalence of certain (2+1)-dimensional integrable spin model and non-linear Shcrodinger equation" *Preprint HEPI* (Alma-Ata, HEPI)

- [22] Myrzakulov R 1996 "Geometry, solitons and the spin description of non-linear evolution equations" *Preprint CNLP-1996-01* (Alma-Ata, CNLP)
- [23] Nugmanova G N 1992 *The Myrzakulov equations: the gauge equivalent counterparts and soliton solutions* (Alma-Ata: KSU)
- [24] Zakharov V E, Takhtajan L A 1979 *TMP* **38** 17
- [25] Belavin A A and Zakharov V E 1978 "Yang-Mills equations as inverse scattering problem" *Phys. Lett. B* **73** 53-57
- [26] Woodhouse N M J and Mason L J 1988 "The Geroch group and non-Hausdorff twistor spaces" *Nonlinearity* **1** 73-114
- [27] Chau L L 1984 "Integrability properties of supersymmetric Yang-Mills fields and relations with other nonlinear systems" in *Group Theoretical Methods in Physics*, proceedings of the XIIIth International Colloquium, ed. W W Zachary, 3-15, World Scientific, Singapore.
- [28] Ward R S 1977 "On self-dual gauge fields" *Phys. Lett. A* **61** 81-82
- [29] Ward R S and Wells R 1990 "Twistor Geometry and Field Theory" *Cambridge University Press, Cambridge, 1990*
- [30] Ward R S 1986 "Multi-dimensional integrable systems" in *Field theory, Quantum Gravity and Strings. II*, proceedings, Meudon and Paris VI, France, 1985/86, eds. H J de Vega and N Sanchez, Lect. Notes Phys. **280** 106-110, Springer-Verlag, Berlin-Heidelberg-New York
- [31] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 "Method for solving the Korteweg-de Vries equation" *Phys. Rev. Lett.* **19** 1095-1097